

The sub- k -domination number of a graph with applications to k -domination

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Abstract

In this paper we introduce and study a new graph invariant derived from the degree sequence of a graph G , called the *sub- k -domination number* and denoted $\text{sub}_k(G)$. We show that $\text{sub}_k(G)$ is a computationally efficient sharp lower bound on the k -domination number of G , and improves on several known lower bounds. We also characterize the sub- k -domination numbers of several families of graphs, provide structural results on sub- k -domination, and explore properties of graphs which are $\text{sub}_k(G)$ -critical with respect to addition and deletion of vertices and edges.

Keywords: sub- k -domination number, k -domination number, degree sequence index strategy

AMS subject classification: 05C69

1 Introduction

Domination is one of the most well-studied and widely applied concepts in graph theory. A set $S \subseteq V(G)$ is *dominating* for a graph G if every vertex of G is either in S , or is adjacent to a vertex in S . A related parameter of interest is the *domination number*, denoted $\gamma(G)$, which is the cardinality of the smallest dominating set of G . Much of the literature on domination is surveyed in the two monographs of Haynes, Hedetniemi, and Slater [11, 12]. For more recent results on domination, see [5, 6, 10, 24] and the references therein.

In 1984, Fink and Jacobson [9] generalized domination by introducing the notion of k -domination and its associated graph invariant, the k -domination number. Given a positive integer k , $S \subseteq V(G)$ is a *k -dominating set* for a graph G if every vertex not in S is adjacent to at least k vertices in S . The minimum cardinality of a k -dominating set of G is the *k -domination number* of G , denoted $\gamma_k(G)$. When $k = 1$, the 1-domination number is precisely the domination number; that is, $\gamma_1(G) = \gamma(G)$. Like domination, k -domination has also been extensively studied; for results on k -domination related to this paper, we refer the reader to [2, 4, 8, 13, 21, 22].

Computing the k -domination number is *NP*-hard [17], and as such, many researchers have sought computationally efficient upper and lower bounds for this parameter. In general, the degree sequence of a graph can be a useful tool for bounding *NP*-hard graph invariants. For example, the *residue* and *annihilation number* of a graph are derived from its degree sequence, and are

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respectively lower and upper bounds on the graph's independence number (cf. [7, 20]). Another example is a lower bound on the domination number due to Slater [23], which will be discussed in the sequel. Recently, Caro and Pepper [1] introduced the *degree sequence index strategy*, or DSI-strategy, which provides a unified framework for using the degree sequence of a graph to bound *NP*-hard invariants. In this paper we introduce a new degree sequence invariant called the sub- k -domination number, which is a sharp lower bound on the k -domination number; our investigation contributes to the known literature on both degree sequence invariants and domination.

Throughout this paper all graphs are simple and finite. Let $G = (V(G), E(G))$ be graph. Two vertices v and w in G are *adjacent*, or *neighbors*, if there exists an edge $vw \in E$. A vertex is an *isolate* if it has no neighbors. The *complement* of G is the graph \overline{G} with the same vertex set, in which two vertices are adjacent if and only if they are not adjacent in G . A set $S \subseteq V(G)$ is *independent* if no two vertices in S are adjacent; the cardinality of the largest independent set in G is denoted $\alpha(G)$. For any edge $e \in E(G)$, $G - e$ denotes the graph G with the edge e removed; For any vertex $v \in V(G)$, $G - v$ denotes the graph G with the vertex v and all edges incident to v removed; for any edge $e \in E(\overline{G})$, $G + e$ denotes the graph G with the edge e added. The *degree* of a vertex v , denoted $d(v)$, is the number of vertices adjacent to v . We will use the notation $n(G) = |V(G)|$ to denote the order of G , $\Delta(G)$ to denote the maximum degree of G , and $\delta(G)$ to denote the minimum degree of G ; when there is no scope for confusion, the dependence on G will be omitted. We will also use d_i to denote the i^{th} element in the *degree sequence* of G , denoted $D(G) = \{\Delta = d_1 \geq d_2 \geq \dots \geq d_n = \delta\}$, which lists the vertex degrees in non-increasing order. We may abbreviate $D(G)$ by only writing distinct degrees, with the number of vertices realizing each degree in superscript. For example, the star $K_{n-1,1}$ may have its degree sequence written as $D(K_{n-1,1}) = \{n-1, 1^{n-1}\}$, and the complete graph K_n may have degree sequence written as $D(K_n) = \{(n-1)^n\}$. For other graph terminology and notation, we will generally follow [15].

This paper is organized as follows. In the next section, we introduce the sub- k -domination number of a graph and show that it is a lower bound on the k -domination number. In Section 3, we characterize the sub- k -domination numbers of several families of graphs and provide other structural results on sub- k -domination. In Section 4, we compare the sub- k -domination number to other known lower bounds on the k -domination number. In Section 5, we explore the properties of $\text{sub}_k(G)$ -critical graphs. We conclude with some final remarks and open questions in Section 6.

2 Sub- k -domination

In this section we introduce the sub- k -domination number of a graph and prove that it is a lower bound on the k -domination number. We first recall a definition and result due to Slater [23], which is a special case of our result. For consistency in terminology, we will refer to Slater's definition as the *sub-domination number* of a graph; this invariant was originally denoted $sl(G)$, and for our purposes will be denoted $\text{sub}(G)$.

Definition 1 ([23]). The *sub-domination number* of a graph G is defined as

$$\text{sub}(G) = \min \left\{ t : t + \sum_{i=1}^t d_i \geq n \right\}.$$

Theorem 1 ([23]). For any graph G , $\gamma(G) \geq \text{sub}(G)$, and this bound is sharp.

For any $k \geq 1$, the k -domination number is monotonically increasing with respect to k ; that is, $\gamma_k(G) \leq \gamma_{k+1}(G)$. Keeping monotonicity in mind, it is natural that a parameter generalizing $\text{sub}(G)$ will need to increase with respect to increasing k . This idea motivates the following definition.

Definition 2. Let $k \geq 1$ be an integer, and G be a graph. The *sub- k -domination number* of G is defined as

$$\text{sub}_k(G) = \min \left\{ t : t + \frac{1}{k} \sum_{i=1}^t d_i \geq n \right\}$$

Since the vertex degrees of G are integers between 0 and $n - 1$, the sorted degree sequence of G can be obtained in $O(n)$ time by counting sort (assuming vertex degrees can be accessed in $O(1)$ time). By maintaining the sum of the first t elements in $D(G)$ and incrementing t , $\text{sub}_k(G)$ can be computed in linear time; we state this formally below.

Observation 2. For any graph G and positive integer k , $\text{sub}_k(G)$ can be computed in $O(n)$ time.

Taking $k = 1$ in Definition 2, we observe $\text{sub}_1(G) = \text{sub}(G)$, and hence $\text{sub}_1(G) \leq \gamma_1(G)$ by Theorem 1. More generally, we will now show that the k -domination number of a graph is bounded below by its sub- k -domination number.

Theorem 3. For any graph G and positive integer k , $\gamma_k(G) \geq \text{sub}_k(G)$, and this bound is sharp.

Proof. Let $S = \{v_1, \dots, v_t\}$ be a minimum k -dominating set of G . By definition, each of the $n - t$ vertices in $V(G) \setminus S$ is adjacent to at least k vertices in S . Thus, the sum of the degrees of the vertices in S , i.e. $\sum_{i=1}^t d(v_i)$, is at least $k(n - t)$. Dividing by k and rearranging, we obtain

$$t + \frac{1}{k} \sum_{i=1}^t d(v_i) \geq n.$$

Since the degree sequence of G is non-increasing, it follows that $\sum_{i=1}^t d_i \geq \sum_{i=1}^t d(v_i)$. Thus,

$$t + \frac{1}{k} \sum_{i=1}^t d_i \geq n. \tag{1}$$

Since $\text{sub}_k(G)$ is the smallest index for which (1) holds, we must have $\text{sub}_k(G) \leq t = \gamma_k(G)$.

When $k = 1$, note that $\text{sub}(K_{n-1,1}) = 1 = \gamma(K_{n-1,1})$. When $k > 1$, let G be a complete bipartite graph with a perfect matching removed where each part of the vertex partition is of size $k + 1$. Then $\text{sub}_k(G) = \min\{t : t + \frac{1}{k} \sum_{i=1}^t k \geq n\} = k + 1 = \gamma_k(G)$. Thus, the bound is sharp for all k . \square

In the next section, we compute $\text{sub}_k(G)$ for several families of graphs and investigate graphs for which $\text{sub}_k(G) = \gamma_k(G)$.

3 Graphs for which $\text{sub}_k(G) = \gamma_k(G)$

In this section we explore the case of equality for Theorem 3. First, note that $\text{sub}(G) = \gamma(G) = n$ for an empty graph G . We therefore exclude empty graphs from the following discussion; that is, assume $\Delta \geq 1$. We begin with two observations for the case $k = 1$.

Proposition 4. *Let G be a graph with $\Delta \geq n - 2$. Then, $\text{sub}(G) = \gamma(G)$.*

Proof. If $\Delta = n - 1$ then $\gamma(G) = 1$ and thus $\text{sub}(G) = \gamma(G)$, since by Theorem 3, $1 \leq \text{sub}(G) \leq \gamma(G) = 1$. If $\Delta = n - 2$, then $\gamma(G) = 2$ since no single vertex can dominate the graph, but a maximum degree vertex and its non-neighbor is a dominating set. Moreover, $\text{sub}(G) \neq 1$ since $1 + (n - 2) < n$; thus, $2 \leq \text{sub}(G) \leq \gamma(G) = 2$. \square

If G is a graph with $\Delta \leq n - 3$, then $\text{sub}(G)$ may not be equal to $\gamma(G)$. For example, let G be the graph obtained by appending a degree one vertex to two leaves of $K_{1,3}$; it can be verified that $\gamma(G) = 3$ and $\text{sub}(G) = 2$.

Proposition 5. *Let G be a graph with $\gamma(G) \leq 2$. Then $\text{sub}(G) = \gamma(G)$.*

Proof. From Theorem 3, if $\gamma(G) = 1$ then $\text{sub}(G) = 1$. Conversely, if $\text{sub}(G) = 1$, then $1 + d_1 \geq n$ and hence from Proposition 4, $\gamma(G) = 1$. Similarly, if $\gamma(G) = 2$ then $\text{sub}(G) \leq 2$; however, since $\text{sub}(G) = 1$ if and only if $\gamma(G) = 1$, it follows that $\text{sub}(G) = 2$. \square

If G is a graph with $\gamma(G) \geq 3$, then $\text{sub}(G)$ may not be equal to $\gamma(G)$. For example, let G be the graph obtained by appending two pendants to each vertex of K_3 ; it can be verified that $\gamma(G) = 3$ and $\text{sub}(G) = 2$.

We next characterize the sub- k -domination number of regular graphs. This will reveal some families of graphs for which $\text{sub}_k(G) = \gamma_k(G)$ for $k \geq 2$.

Theorem 6. *If G is an r -regular graph, then $\text{sub}_k(G) = \lceil \frac{kn}{r+k} \rceil$.*

Proof. Since G is r -regular, $d_i = r$ for $1 \leq i \leq n$. Then, from the definition of sub- k -domination, we have

$$\text{sub}_k(G) + \frac{\text{sub}_k(G)r}{k} = \text{sub}_k(G) + \frac{1}{k} \sum_{i=1}^{\text{sub}_k(G)} d_i \geq n. \quad (2)$$

Rearranging (2), we obtain

$$\frac{kn}{r+k} \leq \text{sub}_k(G). \quad (3)$$

Since $\text{sub}_k(G)$ is the smallest integer that satisfies (3), it follows that $\text{sub}_k(G) = \lceil \frac{kn}{r+k} \rceil$. \square

Note that $\gamma_k(G) = n$ whenever $k > \Delta(G)$. We therefore restrict ourselves to the more interesting case of $k \leq \Delta$. The next example shows an infinite family of graphs for which the sub- k -domination number equals the k -domination number for all $k \leq \Delta$.

Observation 7. *Let C_n be a cycle. For all $k \leq \Delta$, $\text{sub}_k(C_n) = \gamma_k(C_n)$.*

Proof. When $k = 1$, it is known that $\gamma(C_n) = \lceil \frac{n}{3} \rceil$. Since cycles are 2-regular, Theorem 6 gives $\text{sub}(C_n) = \lceil \frac{n}{3} \rceil$. Hence, $\gamma(C_n) = \text{sub}(C_n)$ for all n . When $k = 2$, Theorem 6 gives $\lceil \frac{n}{2} \rceil \leq \text{sub}_2(C_n)$. Since we can produce a 2-dominating set for C_n by first picking any vertex v and adding all vertices whose distance from v is even, it follows that $\gamma_2(C_n) \leq \lceil \frac{n}{2} \rceil$. Thus $\text{sub}_2(C_n) = \gamma_2(C_n)$. \square

As another example, from Proposition 4 and Theorem 6, we see that $\gamma(K_n) = \text{sub}(K_n) = 1$ and $\gamma_2(K_n) = \text{sub}_2(K_n) = 2$ for all n . When $k \geq 3$, $\gamma_k(K_n)$ does not equal $\text{sub}_k(K_n)$ for all n (for example, $\text{sub}_3(K_4) = 2$ but $\gamma_3(K_4) = 3$); however, our next result shows that equality does hold when n is large enough.

Proposition 8. *Let K_n be a complete graph and let $k \leq n - 1$ be a positive integer. Then $\text{sub}_k(K_n) = \gamma_k(K_n) = k$ if and only if $n > (k - 1)^2$.*

Proof. First, note that $\gamma_k(K_n) = k$ for $k \leq n - 1$, since any set of k vertices of K_n is k -dominating, while any set with at most $k - 1$ vertices is at most $(k - 1)$ -dominating. Next, since K_n is regular of degree $n - 1$ it follows from Theorem 6 that

$$\text{sub}_k(K_n) = \left\lceil \frac{kn}{n - 1 + k} \right\rceil \leq k = \gamma_k(K_n).$$

If $\text{sub}_k(K_n) = k$, we must have

$$\frac{kn}{n - 1 + k} > k - 1.$$

Rearranging, we obtain that $n > (k - 1)^2$. □

Our last focus in this section is on the sub- k -domination number and k -domination number of 3-regular, or *cubic*, graphs. First, we recall an upper bound for the k -domination number due to Caro and Roditty [2].

Theorem 9 ([2]). *Let G be a graph, and k and r be positive integers such that $\delta \geq \frac{r+1}{r}k - 1$. Then, $\gamma_k(G) \leq \frac{r}{r+1}n$.*

In particular, for cubic graphs, Theorem 6 and the Caro-Roditty bound (with r taken to be the smallest positive integer satisfying $3 \geq \frac{r+1}{r}k - 1$) imply the following intervals for the k -domination number.

Corollary 10. *Let G be a cubic graph. Then,*

1. $\lceil \frac{n}{4} \rceil \leq \gamma(G) \leq \lfloor \frac{n}{2} \rfloor$,
2. $\lceil \frac{2n}{5} \rceil \leq \gamma_2(G) \leq \lfloor \frac{n}{2} \rfloor$,
3. $\lceil \frac{n}{2} \rceil \leq \gamma_3(G) \leq \lfloor \frac{3n}{4} \rfloor$.

We see from Corollary 10 that $\text{sub}_k(G) = \gamma_k(G)$ for some cubic graphs with small values of n ; for example, $\text{sub}(G) = \gamma(G)$ when $n \leq 6$ and $\text{sub}_2(G) = \gamma_2(G)$ when $n \leq 8$.

4 Comparison to known bounds on $\gamma_k(G)$

A well-known lower bound on the domination number of a graph is $\frac{n}{\Delta+1}$. This bound is not difficult to derive *a priori*, but it immediately follows from the definition of $\text{sub}(G)$ and Theorem 3. In [9], Fink and Jacobson generalized this bound by showing that $\frac{kn}{\Delta+k} \leq \gamma_k(G)$; this also follows from a result of Hansberg and Pepper in [14]. In the following theorem, we show that $\text{sub}_k(G)$ is an improvement on this bound.

Theorem 11. *Let G be a graph; for every positive integer $k \leq \Delta$,*

$$\frac{kn}{\Delta + k} \leq \text{sub}_k(G) \leq \gamma_k(G). \tag{4}$$

Proof. The second inequality in (4) follows from Theorem 3. To prove the first inequality, fix k and let $t = \text{sub}_k(G)$. By definition, $t + \frac{1}{k} \sum_{i=1}^t d_i \geq n$. Since $\Delta \geq d_i$ for $1 \leq i \leq n$, it follows that

$$t + \frac{t\Delta}{k} = t + \frac{1}{k} \sum_{i=1}^t \Delta \geq t + \frac{1}{k} \sum_{i=1}^t d_i \geq n.$$

Rearranging the above inequality gives

$$\frac{kn}{\Delta + k} \leq t = \text{sub}_k(G).$$

□

Recall from Theorem 6 that if G is regular of degree r , then $\text{sub}_k(G) = \lceil \frac{kn}{r+k} \rceil$. Thus, from Theorem 11, we see that regular graphs minimize the sub- k -domination number over all graphs with n vertices and maximum degree Δ . This suggests that in order to maximize the sub- k -domination number, we might consider graphs which are, in some sense, highly irregular with respect to vertex degrees. This motivates the following theorem and its corollary.

Theorem 12. *Let G be a graph; for $1 \leq t \leq \Delta$ let n_t be the number of vertices of G with degree t , let $s_t = \sum_{i=1}^t n_{\Delta+1-i}$, and let $\Delta_t = d_{s_t+1}$. If $s_t + \sum_{i=1}^{s_t} d_i < n$ for some t , then*

$$\frac{kn - \sum_{i=1}^t (\Delta + 1 - \Delta_t - i)n_{\Delta+1-i}}{k + \Delta_t} \leq \text{sub}_k(G).$$

Proof. From the definition of $\text{sub}_k(G)$, we have

$$n \leq \text{sub}_k(G) + \frac{1}{k} \sum_{i=1}^{\text{sub}_k(G)} d_i. \quad (5)$$

Since $s_t + \sum_{i=1}^{s_t} d_i < n$, it follows that $s_t < \text{sub}_1(G) \leq \text{sub}_k(G)$, and thus

$$\sum_{i=1}^{\text{sub}_k(G)} d_i = \sum_{i=1}^{s_t} d_i + \sum_{i=s_t+1}^{\text{sub}_k(G)} d_i. \quad (6)$$

Since $s_t = n_\Delta + n_{\Delta-1} + \dots + n_{\Delta-t+1}$ and since the degree sequence of G is non-increasing and has n_j elements with value j , we have

$$\begin{aligned} \sum_{i=1}^{s_t} d_i &= \Delta n_\Delta + (\Delta - 1)n_{\Delta-1} + \dots + (\Delta - t + 1)n_{\Delta-t+1} \\ &= \sum_{i=1}^t (\Delta + 1 - i)n_{\Delta+1-i}. \end{aligned} \quad (7)$$

Again since $D(G)$ is non-decreasing, we have that $\Delta_t = d_{s_t+1} \geq d_{s_t+2} \geq \dots \geq d_{\text{sub}_k(G)}$. Thus, it follows that

$$\sum_{i=s_t+1}^{\text{sub}_k(G)} d_i \leq \sum_{i=s_t+1}^{\text{sub}_k(G)} \Delta_t = (\text{sub}_k(G) - s_t)\Delta_t. \quad (8)$$

Substituting (6), (7), and (8) into the right-hand-side of (5) yields

$$n \leq \text{sub}_k(G) + \frac{1}{k} \sum_{i=1}^t (\Delta + 1 - i) n_{\Delta+1-i} + \frac{1}{k} (\text{sub}_k(G) - s_t) \Delta_t.$$

By expanding $(\text{sub}_k(G) - s_t) \Delta_t$ and substituting $s_t = \sum_{i=1}^t n_{\Delta+1-i}$, the above inequality can be rewritten as

$$n \leq \text{sub}_k(G) \left(1 + \frac{\Delta_t}{k}\right) + \frac{1}{k} \sum_{i=1}^t (\Delta + 1 - \Delta_t - i) n_{\Delta+1-i}.$$

Rearranging the preceding inequality gives

$$\frac{kn - \sum_{i=1}^t (\Delta + 1 - \Delta_t - i) n_{\Delta+1-i}}{k + \Delta_t} \leq \text{sub}_k(G).$$

□

We note that the bound in Theorem 12 is optimal when t is taken to be the maximum positive integer for which $s_t + \sum_{i=1}^{s_t} d_i < n$. Theorem 12 can be used to give simple lower bounds for the k -domination number of a graph when certain restrictions on the order and maximum degree are met. These bounds also improve on the lower bound given in Theorem 11.

Corollary 13. *Let G be a graph, let n_Δ denote the number of maximum degree vertices of G , and let Δ' denote the second-largest degree of G . If k is a positive integer and $n_\Delta + \frac{\Delta n_\Delta}{k} < n$, then*

$$\frac{kn - n_\Delta(\Delta - \Delta')}{\Delta' + k} \leq \text{sub}_k(G) \leq \gamma_k(G). \quad (9)$$

Proof. Take $t = 1$ in the bound from Theorem 12 and note that $s_1 = n_\Delta$ and $\Delta_1 = d_{n_\Delta+1} = \Delta'$. Since $n_\Delta + \frac{\Delta n_\Delta}{k} < n$, we have that $s_1 + \frac{1}{k} \sum_{i=1}^{s_1} d_i = n_\Delta + \frac{1}{k} \sum_{i=1}^{n_\Delta} d_i = n_\Delta + \frac{\Delta n_\Delta}{k} < n$. Thus, the condition of Theorem 12 is satisfied, and we obtain the first inequality in (9); the second inequality in (9) follows from Theorem 3. □

We see from Corollary 13 that if G has a unique maximum degree vertex, then

$$\frac{kn - \Delta + \Delta'}{\Delta' + k} \leq \gamma_k(G).$$

Corollary 13 gives significant improvements on the lower bound in Theorem 11 whenever the difference between Δ and Δ' is large. For example, consider the corona of $K_{1,n-1}$ ($n \geq 3$) which is obtained by appending a vertex of degree 1 to each of the $n - 1$ vertices of degree 1 in $K_{1,n-1}$. The degree sequence of this graph is $\{n - 1, 2^{n-1}, 1^{n-1}\}$ and its order is $2n - 1$. This graph meets the conditions of Corollary 13, and the bound given in the corollary simplifies to $\frac{(2k-1)n - (k-3)}{2+k}$, whereas the bound given by Theorem 11 is $\frac{k(2n-1)}{n-1+k}$. To compare these two bounds, we first compute the difference between them:

$$\frac{(2k-1)n - (k-2)}{2+k} - \frac{k(2n-1)}{n-1+k} = \frac{(2k-1)n^2 + (4-6k)n + 8k - k^2 - 3}{(2+k)(n-1+k)}.$$

When k is fixed, the difference between these two bounds approaches ∞ as $n \rightarrow \infty$.

5 Critical graphs

There are three natural ways to consider critical graphs in the context of sub- k -domination: graphs which are critical with respect to edge-deletion, edge-addition, and vertex-deletion.

Definition 3. Let G be a graph and k be a positive integer. We will say that

1. G is *edge-deletion-sub $_k(G)$ -critical* if for any $e \in E(G)$, $\text{sub}_k(G - e) > \text{sub}_k(G)$.
2. G is *edge-addition-sub $_k(G)$ -critical* if for any $e \in E(\overline{G})$, $\text{sub}_k(G + e) < \text{sub}_k(G)$.
3. G is *vertex-deletion-sub $_k(G)$ -critical* if for any $v \in V(G)$, $\text{sub}_k(G - v) > \text{sub}_k(G)$.

These properties will respectively be abbreviated as *sub $_k(G)$ -ED-critical*, *sub $_k(G)$ -EA-critical*, and *sub $_k(G)$ -VD-critical*.

In this section, we present several structural results about sub- k -domination critical graphs, including connections to other graph parameters. Throughout the section, we will assume that given a graph G with $V(G) = \{v_1, \dots, v_n\}$ and $D(G) = \{d_1, \dots, d_n\}$ where $d_1 \geq \dots \geq d_n$, it holds that $d_i = d(v_i)$ — in other words, the vertices of G are labeled according to a non-increasing ordering of their degrees.

We first present two results about *sub $_k(G)$ -ED-critical* graphs.

Proposition 14. *Let G be a sub $_k(G)$ -ED-critical graph with $\text{sub}_k(G) = t$. Then $\{v_{t+1}, \dots, v_n\}$ is an independent set of G , and $n - \text{sub}_k(G) \leq \alpha(G)$.*

Proof. Suppose for contradiction that $\{v_{t+1}, \dots, v_n\}$ is not an independent set and let $e = v_x v_y$ be an edge with $v_x, v_y \in \{v_{t+1}, \dots, v_n\}$. Then, the degree sequence of $G - e$ is $d'_1 \geq \dots \geq d'_n$, where $d'_i = d_i$ for all $1 \leq i \leq t$. Thus, $t + \frac{1}{k} \sum_{i=1}^t d'_i = t + \frac{1}{k} \sum_{i=1}^t d_i \geq n$, which implies that $\text{sub}_k(G - e) \leq t$; this contradicts the assumption that G is *sub $_k(G)$ -ED-critical*. Thus, $\{v_{t+1}, \dots, v_n\}$ is an independent set, so $\alpha(G) \geq n - t$. \square

Proposition 15. *Let G be a sub $_k(G)$ -ED-critical graph with no isolates and $\text{sub}_k(G) = t$. Then $\lfloor t + \frac{1}{k} \sum_{i=1}^t d_i \rfloor = n$, and for any $e \in E(G)$, $\text{sub}_k(G - e) = \text{sub}_k(G) + 1$.*

Proof. By definition of *sub $_k(G)$* and since n is an integer, we have that $\lfloor t + \frac{1}{k} \sum_{i=1}^t d_i \rfloor \geq n$. Suppose for contradiction that $\lfloor t + \frac{1}{k} \sum_{i=1}^t d_i \rfloor > n$. Since by Proposition 14, $\{v_{t+1}, \dots, v_n\}$ is an independent set of G and since G has no isolates, we can choose an edge e incident to exactly one vertex in $\{v_1, \dots, v_t\}$. The degree sequence of $G - e$ is $d'_1 \geq \dots \geq d'_n$, where $\sum_{i=1}^t d'_i = (\sum_{i=1}^t d_i) - 1$. Thus,

$$t + \frac{1}{k} \sum_{i=1}^t d'_i = t + \frac{1}{k} \sum_{i=1}^t d_i - \frac{1}{k} \geq \left\lfloor t + \frac{1}{k} \sum_{i=1}^t d_i \right\rfloor - \frac{1}{k} \geq n + 1 - \frac{1}{k} \geq n,$$

meaning $\text{sub}_k(G - e) = t$, which contradicts G being *sub $_k(G)$ -ED-critical*.

Now let e be any edge of G and $d'_1 \geq \dots \geq d'_n$ be the degree sequence of $G - e$. The deletion of e decreases $\sum_{i=1}^{t+1} d_i$ by at most 2, i.e., $\sum_{i=1}^{t+1} d'_i \geq (\sum_{i=1}^{t+1} d_i) - 2$. Thus,

$$(t+1) + \frac{1}{k} \sum_{i=1}^{t+1} d'_i \geq (t+1) + \frac{1}{k} \sum_{i=1}^{t+1} d_i - \frac{2}{k} = t + \frac{1}{k} \left(\sum_{i=1}^t d_i \right) + \frac{d_{t+1} - 2}{k} + 1 \geq n,$$

where in the last inequality $d_{t+1} \geq 1$ since G has no isolates; this implies $\text{sub}_k(G - e) = t + 1 = \text{sub}_k(G) + 1$. \square

Next, we present two analogous results about $\text{sub}_k(G)$ -EA-critical graphs.

Proposition 16. *Let G be a $\text{sub}_k(G)$ -EA-critical graph with $\text{sub}_k(G) = t$. Then the vertices in $\{v \in V(G) : d(v) < d_t\}$ form a clique.*

Proof. Suppose on the contrary that there are two non-adjacent vertices v_x and v_y with $d_t > d_x \geq d_y$. Then, the degree sequence of $G + v_x v_y$ is $d'_1 \geq \dots \geq d'_n$, where $d'_i = d_i$ for all $1 \leq i \leq t$. This implies that $\text{sub}_k(G + e) = \text{sub}_k(G)$, a contradiction. \square

Proposition 17. *Let G be a $\text{sub}_k(G)$ -EA-critical graph with no isolates and $\text{sub}_k(G) = t$. Then, for each $e \in E(\overline{G})$, $\text{sub}_k(G + e) = \text{sub}_k(G) - 1$.*

Proof. Let e be any edge in \overline{G} and $d'_1 \geq \dots \geq d'_n$ be the degree sequence of $G + e$. The addition of e increases $\sum_{i=1}^t d_i$ by at most 2, i.e., $\sum_{i=1}^t d'_i \leq (\sum_{i=1}^t d_i) + 2$. Thus,

$$(t - 2) + \frac{1}{k} \sum_{i=1}^{t-2} d'_i = (t - 2) + \frac{1}{k} \sum_{i=1}^t d'_i - \frac{d'_t + d'_{t-1}}{k} \leq t + \frac{1}{k} \sum_{i=1}^t d_i - \frac{d'_t + d'_{t-1}}{k} \leq n - \frac{d'_t + d'_{t-1}}{k} < n,$$

where in the last inequality $\frac{d'_t + d'_{t-1}}{k} > 0$ since G has no isolates; this implies $\text{sub}_k(G + e) > t - 2$, so $\text{sub}_k(G + e) = \text{sub}_k(G) - 1$. \square

Graphs that are $\text{sub}_k(G)$ -VD-critical differ from $\text{sub}_k(G)$ -ED-critical graphs and $\text{sub}_k(G)$ -EA-critical graphs, in the sense that it is possible for $\text{sub}_k(G - v)$ and $\text{sub}_k(G)$ to differ by much more than 1. For example, this is the case for the star $K_{n-1,1}$ when the center of the star is the vertex removed. We now show another result for $\text{sub}_k(G)$ -VD-critical graphs.

Proposition 18. *Let G be a $\text{sub}_k(G)$ -VD-critical graph with $\text{sub}_k(G) = t$. Then each vertex in $\{v_{t+1}, \dots, v_n\}$ is adjacent to at least $k + 1$ vertices in $\{v_1, \dots, v_t\}$.*

Proof. Suppose that $v_x \in \{v_{t+1}, \dots, v_n\}$ is adjacent to at most k vertices in $\{v_1, \dots, v_t\}$. Then $G - v_x$ has degree sequence d'_1, \dots, d'_{n-1} such that $\sum_{i=1}^t d'_i \geq (\sum_{i=1}^t d_i) - k$. Thus, $t + \frac{1}{k} \sum_{i=1}^t d'_i \geq t + \frac{1}{k} \sum_{i=1}^t d_i - 1 \geq n - 1$, which implies that $\text{sub}_k(G - v_x) \leq t$; this contradicts the assumption that G is $\text{sub}_k(G)$ -VD-critical. \square

6 Conclusion

In this paper, we introduced the sub- k -domination number and showed that it is a computationally efficient lower bound on the k -domination number of a graph. We also showed that the sub- k -domination number improves on several known bounds for the k -domination number, and gave some conditions which assure that $\text{sub}_k(G) = \gamma_k(G)$. This investigation was a step toward the following general question:

Problem 1. *For each positive integer k , characterize all graphs for which $\gamma_k(G) = \text{sub}_k(G)$.*

As another direction for future work, it would be interesting to define and study an analogue of sub- k -domination which is an upper bound to the k -domination number, or explore degree sequence based invariants which bound the connected domination number or the independent domination number of a graph.

Acknowledgements

This work is supported by the National Science Foundation, Grant No. 1450681 (B. Brimkov).

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